§11.1 Sequences.

Definition: Sequence: A LIST of numbers. Notation: $a_1, a_2, a_3, \cdots, a_n, \cdots$

a_n is called the n^{th} term of the sequence. $(n=1,2,3,\cdots\cdots)$

Eg. 0 (Motivation). Investment at bank: \$1,000 is deposited into bank at 6% annual interest rate.

Find the money in the account AT THE END of n^{th} year.

After 1 year, $1000 + \underbrace{1000 \times 6\%}_{\text{interest}} = 1000 \times (1+6\%) = 1000 \times 1.06$ $\quad a_1$

After 2 years, $[1000 \times 1.06] + [1000 \times 1.06] \times 6\% = \cdots\cdots = [1000 \times 1.06] \times 1.06$ $\quad a_2$

$\cdots\cdots\cdots$

After n years, $\cdots\cdots = 1000 \times 1.06 \times \cdots \times 1.06$ $\boxed{= 1000 \times 1.06^n \quad a_n.}$

Goals of this section (key points in exam):

I: Find a formula of the n^{th} term a_n given the first several terms

II: Find the limit of a sequence a_n (Determine whether a_n is CONV or DIV).

Remarks:

① $a_1, a_2, \cdots, a_n, \cdots$ can also be denoted as $\{a_n\}_{n=1}^{\infty}$, $\{a_n\}$, or $\{a_1, a_2, \cdots, a_n, \cdots\}$.

② Sequence Vs Function. a_n VS $f(x)$. One can always replace n in the formula of a_n by x to get a function $f(x)$ in the usual sense and vice verse.

$a_n \rightleftarrows f(x)$. $\quad a_n = n \cdot e^{-n} \xrightarrow{\text{Can be viewed as}} f(x) = x \cdot e^{-x}$

③. LIST VS Collection. By 'list of numbers', we mean the numbers are given in a DEFINITE ORDER, which is labeled by index $n=1,2,\cdots$.
So we have the so-called '1st term a_1', '2nd term a_2', ... 'n^{th} term a_n'.

④ Trivial sequence: Constant sequence: All a_n are the same. $\{0,0,0,\cdots,0,\cdots\}$, $a_n=0$.
$\{2,2,2,\cdots,2,\cdots\}$ $a_n=2$

I: Formula for a_n.

e.g.1. Consider the following sequence: $9, 25, 49, 81, \cdots\cdots$

Write a formula for the n^{th} term of this sequence, where $n=1,2,3,\cdots$.

Hint: $9 = 3^2 = (2\cdot 1+1)^2$, $25 = 5^2 = (2\cdot 2+1)^2$, $49 = 7^2 = (2\cdot 3+1)^2$, $81 = 9^2 = (2\cdot 4+1)^2$

Solution: $a_n = (2n+1)^2$, $n=1,2,3,\cdots$

Remark: Always double check your answer by pluging in $n=1,2,3,4$ after you get $(2n+1)^2$.

e.g.2. The sequence $-1, +1, -1, +1, -1, \cdots$ has formula $a_n = (-1)^n$, $n=1,2,\cdots$

The sequence $+1, -1, +1, -1, +1, \cdots$ has formula $a_n = (-1)^{n+1}$, $n=1,2,\cdots$.

and also $a_n = (-1)^{n-1}$, $n=1,2,\cdots$ (Both are correct

★ e.g.3. Find the formula for the sequence $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \cdots\cdots$ $n=1,2,3,\cdots$

and then for the sequence $-1, +\frac{1}{2}, -\frac{1}{4}, +\frac{1}{8}, -\frac{1}{16}, \cdots$ $n=1,2,3,\cdots$

Solution: Notice that $1 = \frac{1}{2^0}$. Therefore, $a_n = \frac{1}{2^{n-1}}$, $n=1,2,3,\cdots$ is the formula.

(Double check: $a_1 = \frac{1}{2^{1-1}} = \frac{1}{2^0} = 1$, $a_2 = \frac{1}{2^{2-1}} = \frac{1}{2}$, $a_3 = \frac{1}{2^{3-1}} = \frac{1}{2^2}$, $\cdots$)

For the second sequence, it is a produce of e.g.2 and the first sequence. ~~therefore~~

i.e., $-1 = (-1)\cdot\frac{1}{2^0}$, $\frac{1}{2} = (+1)\cdot\frac{1}{2^1}$, $-\frac{1}{4} = (-1)\cdot\frac{1}{2^2}$, $\cdots\cdots$

Therefore, $a_n = (-1)^n\cdot\frac{1}{2^{n-1}}$, $n=1,2,3,\cdots\cdots$

Remark: Check the following identity holds: $\boxed{(-1)^n\frac{1}{2^{n-1}}} = -\frac{1}{(-2)^{n-1}} = \boxed{2\cdot\left(-\frac{1}{2}\right)^n}$

Hint: use the following properties: $x^{a+b} = x^a\cdot x^b$, $(a\cdot b)^x = a^x\cdot b^x$.

Some frequently used sequences: $a_n = (-1)^n$, $a_n = (-1)^{n+1}$, $a_n = (-1)^{n-1}$

ODD sequence (from 1): • $a_n = 2n-1$, $n=1,2,\cdots$, $\{1,3,5,7,\cdots\}$

EVEN sequence (from 2): • $a_n = 2n$, $n=1,2,\cdots$ $\{2,4,6,8,\cdots\}$

Exponential sequence: e.g. • $a_n = 2^n$, • $a_n = \frac{1}{3^{n-1}}$ • $a_n = \left(\frac{4}{5}\right)^n$

(Also called Geometric sequence, which will be discussed more §11.2).

II: Limit of a_n as $n \to +\infty$.

If $\lim_{n\to\infty} a_n$ exists, the sequence a_n is called a '$\boxed{\text{convergent}}$' sequence. Otherwise, we say the sequence a_n is $\boxed{\text{divergent}}$.

e.g.4. constant sequence always converges. $a_n = 5, n=1,2,\dots$ $\boxed{\lim_{n\to\infty} a_n = 5}$ conv.

e.g.5. Consider the sequence $a_n = n\cdot e^{-n}, n=1,2,\dots$. Find the limit of this sequence.

Hint: It is equivalent to ask you to find $\lim_{x\to\infty} x\cdot e^{-x}$ as we learned in $\boxed{\S 6.8}$.

Solution: $a_n = n\cdot e^{-n}$. replace n by $+\infty$, we have $\infty\cdot e^{-\infty}$ $(e^{-\infty}=0)$ indeterminate type

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\cdot e^{-n} = \lim_{n\to\infty}\frac{n}{e^n} \overset{\text{l'Hosp}}{=\!=} \lim_{n\to\infty}\frac{n'}{(e^n)'}$$

n' and $(e^n)'$ are the derivatives of n and e^n with respect n. $n'=1$, $(e^n)'=e^n$

$$= \lim_{n\to\infty}\frac{1}{e^n}$$

$$\boxed{= \frac{1}{\infty} = 0}$$

Conclusion, the sequence $a_n = n\cdot e^{-n}$ converges to 0, i.e. $\lim_{n\to\infty} a_n = 0$

e.g.6. Evaluate $\lim_{n\to\infty}\frac{-2n^3+1}{n^2+3n}$, determine whether the sequence is Conv or DIV.

Solution: $$\overset{\text{l'Hosp}}{=\!=} \lim_{n\to\infty}\frac{(-2n^3+1)'}{(n^2+3n)'} = \lim_{n\to\infty}\frac{-2\cdot 3n^2}{2n+3} \overset{\text{l'Hosp}}{=\!=} \lim_{n\to\infty}\frac{-2\cdot 3\cdot 2n}{2}$$

$$= \lim_{n\to\infty} -6\cdot n = \boxed{-\infty}.$$

The sequence $\frac{-2n^3+1}{n^2+3n}$ is DIVERGENT.

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e.g.7. $a_n = (-1)^n, n=1,2,\dots$, $\{-1,1,-1,\dots\}$ The sequence jumps from 1 to -1 forever so doesnot have a limit, i.e. $\boxed{(-1)^n \text{ diverges}}$. So does $(-1)^{n+1}$.

e.g.8. Consider $a_n = (-1)^n\cdot\frac{1}{2^{n-1}}$ (in eg.3.), $-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \dots$

Although the sign changes from + to − ~~alternatively~~ alternatively, the denominator 2^{n-1}

$2^{n-1} \xrightarrow{n\to\infty} \infty$ (tends to ∞ as n goes to ∞). Therefore,

$$\lim_{n\to\infty} (-1)^n\cdot\frac{1}{2^{n-1}} = \pm\frac{1}{2^\infty} = \pm\frac{1}{\infty} = 0.$$ Converges to 0.

Review of some frequently used limit techniques from Cal I and §6.8 Cal II.

① $\boxed{\text{Leading Term Trick}}$ for $\frac{\text{Polynomial}}{\text{Polynomial}}$ type. (only n to some power is involved).

Rule: Find the terms with highest order (power) in n. Drop all the rest terms, simplify the expression and then take the limit as $n \to \infty$.

$$\lim_{n\to\infty} \frac{6-2n+n^4}{5n^4-3n^2+100} = \lim_{n\to\infty} \frac{\cancel{6-2n}+n^4}{5n^4 \cancel{-3n^2+100}}$$

← leading term in the numerator

leading term in the denominator.

$$= \lim_{n\to\infty} \frac{\cancel{n^4}}{5\cancel{n^4}} \xrightarrow{\text{simplify}} \boxed{\frac{1}{5}} \quad \text{conv}$$

$$\lim_{n\to\infty} \frac{-3n^2+5n}{1000\cdot n+5\cdot n^{2.5}+6} = \lim_{n\to\infty} \frac{-3n^2}{5\cdot n^{2.5}} = \lim_{n\to\infty} \frac{-3}{5n^{0.5}} = \lim_{n\to\infty} \frac{-3}{5\sqrt{n}} = \frac{-3}{5\cdot\infty} = \boxed{0}$$

Remark: Compare these two examples with $\boxed{\text{e.g. 6}}$ and the $\boxed{\text{l'Hospital rule}}$.

② $\boxed{a^{+\infty} = +\infty \text{ if } a>1}$ and $\boxed{a^{+\infty} = 0 \text{ if } 0<a<1}$.

Remember the limits via the following examples:

$$\lim_{n\to\infty} 2^{n-1} = 2^{\infty} = \infty, \quad \lim_{n\to\infty} [1.01]^n = 1.01^{\infty} = \infty; \quad \lim_{n\to\infty} \left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^{\infty} = 0$$

For negative numbers, use the trick $\boxed{(-a)^n = (-1)^n \cdot a^n}$. $(-5)^n = (-1)^n \cdot 5^n$

③ Exponential l'Hospital Trick (Please refer to lec-notes Week 4, Page 7, e.g. 4 in §6.8)
(ww11, 12 are relied on this)

Evaluate $\lim_{n\to\infty} 3\cdot\left(1+\frac{1}{n^2}\right)^{5n}$

It is enough to consider the limit for

$$\left(1+\frac{1}{n^2}\right)^{5n} = e^{\ln\left(1+\frac{1}{n^2}\right)^{5n}}$$

Step 1: $\ln\left(1+\frac{1}{n^2}\right)^{5n} = 5n\cdot\ln\left(1+\frac{1}{n^2}\right)$

$$= \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{5n}}$$

Step 2:

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{5n}}$$

$$\stackrel{l'H}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n^2}}\cdot\frac{-2}{n^3}}{\frac{-1}{5n^2}}$$

$$= \lim_{n\to\infty} \frac{10\cdot n^2}{n^3+\cancel{n}}$$

$= 0$ (leading term Rule)

Step 3:

$$\lim_{n\to\infty} 3\left(1+\frac{1}{n^2}\right)^{5n}$$

$$= 3\cdot\lim_{n\to\infty} e^{\boxed{\ln\left(1+\frac{1}{n^2}\right)^{5n}}}$$

$= 3\cdot e^{0}$ ← results from Step 2

$= \boxed{3}$

§11.2 Series

Key points:

① Definition and properties of Series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$

②. TEST FOR DIVERGENCE (DIV TEST). $\lim_{n\to\infty} a_n \neq 0$ implies $\sum_{n=1}^{\infty} a_n$ DIV.
(Also called nth ~~term~~ term test for divergence)

③ Geometric Series: $a + a\cdot r + a\cdot r^2 + a\cdot r^3 + \cdots + a\cdot r^{n-1} + \cdots = \frac{a}{1-r}$, $|r|<1$.

- Motivation: (Sigma Notation) (Remark: For a complete review of sigma notation, refer to Appendix E in the textbook).

eg.0. Expand the following SIGMA NOTATION as a usual sum.

$$\sum_{n=0}^{4} \frac{1}{2^n} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}; \quad \sum_{n=2}^{5} \frac{1}{n-1} = \frac{1}{2-1} + \frac{1}{3-1} + \frac{1}{4-1} + \frac{1}{5-1}.$$

$$\sum_{i=1}^{N} a_i = a_1 + a_2 + \cdots + a_{N-2} + a_{N-1} + a_N; \quad \sum_{k=1}^{100} 3 = \underbrace{3+3+\cdots+3+3}_{100} \; (= 3\cdot 100 = 300)$$

Question: what happens if we put ∞ on the top of $\sum$? (i.e. $\sum_{n=1}^{\infty}$)

- $\boxed{\text{Definition:}}$ Give a sequence $\{a_n\}_{n=1}^{\infty}$ (i.e. $a_1, a_2, a_3, \cdots, a_n, \cdots$). We call the INFINITE SUM of all a_n an infinite SERIES: $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$.
And denote by $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$ for short.

- eg.1. (Two trivial examples)

$$\sum_{n=0}^{\infty} 0 = 0 + 0 + 0 + \cdots + 0 + \cdots \; (= 0); \quad \sum_{n=1}^{\infty} 3 = 3 + 3 + \cdots + 3 + \cdots \; (= \infty, \text{ we may guess})$$

- We are interested in whether the infinite sum adds up to some finite number or infinity. Roughly speaking, if $\sum_{n=1}^{\infty} a_n$ adds up to some (fixed) finite number, we say $\boxed{\text{the series is CONVERGENT}}$. Otherwise, $\boxed{\sum_{n=1}^{\infty} a_n \text{ diverges}}$.

- Remark: There are only FEW examples we can compute the exact sum of the series. All the rest of Chapter 11 are methods to determine CONV/DIV without knowing the exact sum.

- Linear properties of Series: $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are two sequences, C is a constant.

$\sum_{n=1}^{\infty}(C a_n) = C\left(\sum_{n=1}^{\infty} a_n\right)$; $\sum_{n=1}^{\infty}(a_n+b_n) = \left(\sum_{n=1}^{\infty} a_n\right)+\left(\sum_{n=1}^{\infty} b_n\right)$; $\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n = \cdots = a_1+a_2+a_3+\sum_{n=4}^{\infty} a_n = \cdots$

- $\boxed{\text{TEST FOR DIVERGENCE. (DIV TEST)}}$ (Also called "n^{th} term test" for short)

Theorem: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is DIVERGENT. i.e., if the sequence a_n does not have a limit $\boxed{\text{or}}$ the limit is NOT zero, then the corresponding series $\sum_{n=1}^{\infty} a_n$ is divergent.

e.g. 2. Fill in the table. Determine whether the following sequences and series are Conv or DIV.

a_n	$(-1)^n$	$2+\sin(\frac{1}{n})$	$\frac{2n+3}{2n+1}$	$\left(\frac{3}{2}\right)^{n-1}$
$\lim_{n\to\infty} a_n$.	D.N.E. (± 1)	$2+\sin 0 = 2$	$\frac{2}{2}=1$	∞ (since $\frac{3}{2}>1$)
a_n is Conv or DIV	DIV	CONV	CONV	DIV
$\sum_{n=1}^{\infty} a_n$ is Conv or DIV.	DIV	DIV	DIV	DIV

Remark: If $\lim_{n\to\infty} a_n = 0$, DIV TEST is inconclusive. We need to move on to methods in the following section

- One particular series (sequence) we can compute $\boxed{\text{the exact sum}}$.

Geometric Series. (G.S.): $a_n = a \cdot r^{n-1}$, $n=1,2,3,\cdots$. a, r are two constants

$a + a\cdot r + a\cdot r^2 + \cdots$, which can be written as $\boxed{\sum_{n=1}^{\infty} a\cdot r^{n-1} = \sum_{n=0}^{\infty} a\cdot r^n}$.

a is called the FIRST TERM of the series. r is called COMMON RATIO.

- Conclusion on CONV/DIV of G.S.

If $|r| \geq 1$, $\sum_{n=1}^{\infty} a\cdot r^{n-1}$ is divergent. If $|r|<1$, $\sum_{n=1}^{\infty} a\cdot r^{n-1}$ is convergent AND.

$$\boxed{\sum_{n=1}^{\infty} a\cdot r^{n-1} = \frac{a}{1-r}} \left(= \frac{\text{FIRST TERM}}{1 - \text{COMMON RATIO}}\right)$$

e.g. 3. Consider $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^{n-1}$ in e.g. 2. This is a G.S. where $a=1$, $r=\frac{3}{2}$

$|r|=\left|\frac{3}{2}\right|>1 \Rightarrow \sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^{n-1}$ diverges.

☆ e.g. 4 (s16, 15pts) Find the sum of the series $\sum_{n=0}^{\infty}\frac{2-3^{n+1}}{6^n}$

Solution: Step 1: (linear properties) $=\sum_{n=0}^{\infty}\frac{2}{6^n} - \sum_{n=0}^{\infty}\frac{3^{n+1}}{6^n}$ (break down the original series into TWO G.S.).

Step 2: G.S. 1 $\sum_{n=0}^{\infty}\frac{2}{6^n}$ (standard G.S. $a=2$, $r=\frac{1}{6}$)

Hint: $\frac{2}{6^n}=2\cdot\left(\frac{1}{6}\right)^n$, $\sum_{n=0}^{\infty}\frac{2}{6^n}=\underset{\overset{\uparrow}{a}}{\frac{2}{6^0}}+2\cdot\underset{\overset{\uparrow}{r}}{\frac{1}{6^1}}+2\cdot\frac{1}{6^2}+\cdots$

$$= \frac{a}{1-r}$$

$$= \frac{2}{1-\frac{1}{6}} = 2\cdot\frac{6}{5}=\frac{12}{5}$$

Step 3: G.S. 2. $\sum_{n=0}^{\infty}\frac{3^{n+1}}{6^n}=\sum_{n=0}^{\infty}\frac{3^1\cdot 3^n}{6^n}=\sum_{n=0}^{\infty}3\cdot\left(\frac{3}{6}\right)^n=\sum_{n=0}^{\infty}\underset{\overset{\uparrow}{a}}{3}\cdot\underset{\overset{\uparrow}{r}}{\left(\frac{1}{2}\right)}^n$

$$= \frac{a}{1-r}, \quad a=3,\ r=\frac{1}{2}$$

$\left(=3+3\cdot\frac{1}{2}+3\cdot\frac{1}{2^2}+\cdots\right)$

$$= \frac{3}{1-\frac{1}{2}} = 6$$

Step 4: $\sum_{n=0}^{\infty}\frac{2-3^{n+1}}{6^n}=\sum_{n=0}^{\infty}\frac{2}{6^n}-\sum_{n=0}^{\infty}\frac{3^{n+1}}{6^n}=\frac{12}{5}-6=\boxed{-\frac{18}{5}}$.

Question: consider $\sum_{n=1}^{\infty}\frac{2-3^{n+1}}{6^n}$ instead. What will be different compared to e.g. 4?

Hints for Webwork:

- Common trick for G.S.: $\frac{(-1)^n}{a^n}=\left(-\frac{1}{a}\right)^n$. $\square^{n+1}=\square^n\cdot\square^1$, $\bigcirc^{n-1}=\bigcirc^n\cdot\bigcirc^{-1}=\frac{\bigcirc^n}{\bigcirc}$
- ww 4: $\left(\frac{1}{7}\right)^{\frac{n}{2}}=\left(\frac{1}{7}\right)^{\frac{1}{2}\times n}=\left[\left(\frac{1}{7}\right)^{\frac{1}{2}}\right]^n=\left[\frac{1}{7^{\frac{1}{2}}}\right]^n=\left(\frac{1}{\sqrt{7}}\right)^n$
- ww 6: (Represent a repeating decimal as Geometric series.)

  $0.\overline{1}=0.11111\ldots = 0.1+0.01+0.001+0.0001+\cdots+0.0\cdots01+\cdots$

  $=\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+\cdots+\cdots+\cdots$
- ww 7, 8, 9. consider x as a fixed parameter. Find r in terms of x, $|r|<1$.
- ww 10. DIV TEST. - Exponential l'Hospital Trick.

§11.3 The integral test and P-Series.

key points

- Series $\sum_{n=1}^{\infty} a_n \longleftrightarrow \int_1^{\infty} f(x)dx$ improper integral, conv/DIV simultaneously.
- p-Series: $\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{conv} & p>1 \\ \text{DIV} & p\leq 1 \end{cases}$

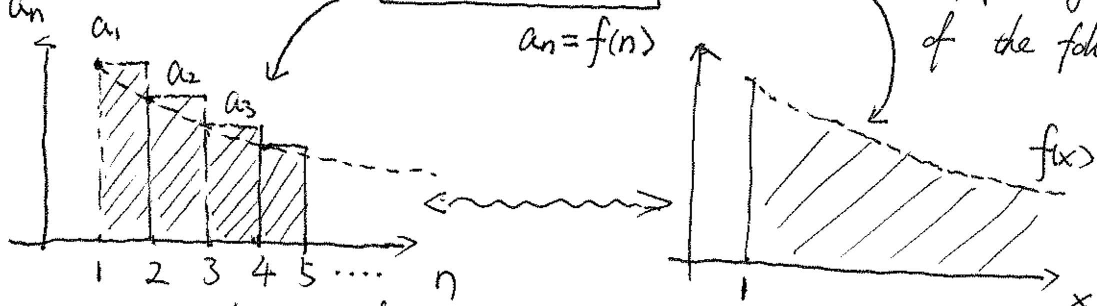
- Integral Test: Give $\sum_{n=1}^{\infty} a_n$. Rewrite a_n as $f(x)$. (Replace n by x in the formula of a_n). Assume that $f(x)$, i.e. ($f(n)=a_n$), is continuous, positive and DECREASING. We have ① If $\int_1^{+\infty} f(x)dx$ conv, then $\sum_{n=1}^{\infty} a_n$ conv. ② If $\int_1^{+\infty} f(x)dx$ DIV, then $\sum_{n=1}^{\infty} a_n$ DIV.

Motivation/Ideas of Integral Tests

$$\sum_{n=1}^{\infty} a_n \sim \int_1^{+\infty} f(x)\,dx.$$

$a_n = f(n)$

Improper integral represents the area of the following region.



Series represents the sum of the area of the rectangles: $a_1 \cdot 1 + a_2 \cdot 1 + a_3 \cdot 1 + \cdots$

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e.g.① Q: Can integral test be used to determine the conv/DIV of $\sum_{n=1}^{\infty} \sin(n)$?
(S16, 6pts). Answer: $\sin(n) \longleftrightarrow \sin x = f(x)$ is NOT positive and DECREASING. The integral test hypotheses ARE NOT met, so it cannot be applied.

Remark: $\lim_{n\to\infty} \sin(n)$ DOES NOT EXIST. So $\sum_{n=1}^{\infty} \sin(n)$ diverges due to DIV TEST.

- (Trivial) Example of Integral Test: $\sum_{n=1}^{\infty} \frac{1}{n}$, $a_n = \frac{1}{n}$, $f(x) = \frac{1}{x}$ continuous, positive and decreasing.

$$\int_1^{\infty} f(x)dx = \int_1^{+\infty} \frac{1}{x}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}dx = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty} \ln|t| = \ln(+\infty) = +\infty.$$

$\int_1^{+\infty} f(x)dx$ diverges, & therefore, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

e.g. 1. Determine whether the following series converges or diverges. State the test you use.

(s16, 10pts) $\sum_{n=2}^{\infty} \frac{5}{n\ln n}$. $\left(a_n = \frac{5}{n\ln n} \longleftrightarrow f(x) = \frac{5}{x\ln x} \text{ satisfies } f(n) = a_n. \right)$

Solution: Since $f(x) = \frac{5}{x\ln x}$ is continuous, positive and decreasing for $n \geq 2$, the integral test can be applied.

$$\int_2^{\infty} \frac{5}{x\ln x}\,dx = \lim_{t\to\infty} \int_2^t \frac{5}{x\ln x}\,dx$$

Hint: $(\ln x, \frac{1}{x})$ pair, u-sub: $\underline{u = \ln x}$, $du = \frac{1}{x}dx$

$$= \lim_{t\to\infty} \int_2^t \frac{5}{\ln x} \cdot \frac{1}{x}dx$$

$$= \lim_{t\to\infty} \int \frac{5}{u} \cdot du = \lim_{t\to\infty} 5\ln|u| = \lim_{t\to\infty} 5\ln|\ln x| \Big|_{x=2}^{x=t}$$

$$= \lim_{t\to\infty} 5\ln|\ln t| - 5\ln|\ln 2|$$

$$= \infty \quad (\text{Hint: } \ln\ln\infty = \ln\infty = \infty)$$

Therefore, $\int_2^{\infty} \frac{5}{x\ln x}\,dx$ diverges.

$\boxed{\text{So the series } \sum_{n=2}^{\infty} \frac{5}{n\ln n} \text{ diverges}}$. ※

Remark: In general, similar argument works for $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ with different parameters $p = 1, 2, 3, \ldots$ (ww 3, 4)

- Series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called p-series with parameter p (any number)

  Conclusion of ConV/DIV: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

e.g. 2. Determine whether the following series is ConV or DIV.

- $\sum_{n=7}^{\infty} \frac{2}{n^3}$. p-series $p = 3 > 1$ ConV.
- $\sum_{n=1}^{\infty} -5n^{-\frac{1}{3}}$, $a_n = -5 \cdot n^{-\frac{1}{3}} = -5 \cdot \frac{1}{n^{\frac{1}{3}}}$, p-series, $p = \frac{1}{3} < 1$, DIV.
- $\sum_{n=2}^{\infty} \frac{4}{\sqrt{4n^e}}$. $a_n = \frac{4}{\sqrt{4n^e}} = \frac{4}{\sqrt{4} \cdot \sqrt{n^e}} = \frac{4}{2 \cdot n^{\frac{e}{2}}}$, $p = \frac{e}{2} > 1$, ConV.
- $\sum_{n=1}^{\infty} \frac{3\sqrt{n}}{n}$, $a_n = \frac{3\sqrt{n}}{n} = \frac{3}{n^{\frac{1}{2}}}$, $p = \frac{1}{2} < 1$, DIV.

- Proof of conclusion on p-series via integral test. (Two typical p-values).

p-value	$p=2 \quad (p>2)$	$p=\frac{1}{2} \quad (p<\frac{1}{2})$
p-series	$\sum_{n=1}^{\infty}\frac{1}{n^2}$	$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$ (or $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$)
Improper integral	$\int_1^{+\infty}\frac{1}{x^2}\,dx$	$\int_1^{+\infty}\frac{1}{x^{\frac{1}{2}}}\,dx$.
Test for the integral	$=\lim_{t\to\infty}\int_1^t x^{-2}\,dx$. $=\lim_{t\to\infty} -x^{-1}\Big\vert_1^t$ $=\lim_{t\to\infty} -\frac{1}{t}+1=\boxed{1}$	$=\lim_{t\to\infty}\int_1^t x^{-\frac{1}{2}}\,dx$. Hint: $\int x^{-\frac{1}{2}}=\frac{1}{-\frac{1}{2}+1}x^{-\frac{1}{2}+1}=2\cdot x^{\frac{1}{2}}$ $=\lim_{t\to\infty} 2\sqrt{x}\Big\vert_1^t$ $=\lim_{t\to\infty} 2\sqrt{t}-2\sqrt{1}=\boxed{+\infty}$
Conclusion:	$\int_1^{+\infty}\frac{1}{x^2}dx$ conv $\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2}$ conv.	$\int_1^{+\infty}\frac{1}{x^{\frac{1}{2}}}dx$ DIV $\Rightarrow \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ DIV.

More remarks:

- p-series VS Geometric Series:

$\sum_{n=1}^{\infty}\frac{1}{n^p}$ $\qquad$ $\sum_{n=1}^{\infty}a\cdot r^{n-1}=\sum_{n=0}^{\infty}a\cdot r^n$

p, r / condition	p-Series.	G.S.
Convergent	$p>1$	$\lvert r\rvert<1$
Divergent.	$p\leq 1$	$\lvert r\rvert\geq 1$.

G.S. has exact value which can be computed via $\sum_{n=0}^{\infty}a\cdot r^n=\frac{a}{1-r}$, $\lvert r\rvert<1$

p-Series ('s exact value) can not be evaluated. The value of improper integral is NOT equal to the sum of p-Series.

$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{2}{3}}}$ VS $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$,

$p=\frac{2}{3}<1$, $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{2}{3}}}$ DIV.

$r=\frac{2}{3}<1$, $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ conv. $\left(=\frac{\frac{2}{3}}{1-\frac{2}{3}}=2\right)$

$\left(a=\frac{2}{3},\ r=\frac{2}{3}.\right)$

- In the following sections, we will consider series which looks similar either to p-series (such as $\sum_{n=1}^{\infty}\frac{1}{(2n+1)^3}$, $\sum_{n=2}^{\infty}\frac{n^2-1}{3n^3+1}$) or a geometric series (such as $\sum\frac{1}{(3^n+7)}$), or a combination of these two (such as $\sum\frac{n^2}{7^n}$).